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# EFFECTIVE PROPERTIES OF SMART ELASTIC LAMINATES AND THE SCREENING PHENOMENON

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**Abstract**—Non-stationary phenomena arising in structures can be effectively controlled via smart (intelligent) materials. The corresponding mathematical problems contain controls in the coefficients of hyperbolic equations, these controls depending both on position and time. When appropriately activated, such controls may provide 100% screening of some extended parts of the structure from the invasion of dynamic disturbances. For bars exposed to longitudinal vibrations, the material pattern producing such a screening effect can be the combination of rank-one laminar composites in space-time. Such a composite is described as an array of alternating segments with different pairs $(\rho_1, k_1)$ and $(\rho_2, k_2)$ of density and stiffness of smart material, this whole array traveling along the bar with a suitable constant speed $V$. © 1997 Elsevier Science Ltd. All rights reserved.

## 1. INTRODUCTION

The appearance of smart materials has made it realistic to claim for the structures properties that may vary both in space and time, thus making these structures highly responsive to the non-stationary environment. We shall place emphasis on the design of material patterns able to block the propagation of undesirable disturbances caused by impulsive loads, impacts, etc., into some extended parts of the structure vulnerable to a material damage. Such screening may be achieved through the deployment of special assemblages of smart materials, e.g., smart laminar composites, activated at appropriate location at appropriate time.

Mathematically, the problem reduces to that of the control of coefficients in hyperbolic equations both in space and time. This control is assumed to be materialized through the sensors and actuators distributed throughout a smart structure.

## 2. BASIC CONCEPTUAL FEATURES OF SMART MATERIAL DESIGN

The above design concept can be efficiently implemented through the use of smart materials. A smart material is assumed to have the capacity to change its properties *both in space and time* to suit the demands placed upon it by the environment.

The smart material is constructed using a distributed set of sensors and actuators embedded within the body of a structure. The optimal design of structures based on use of smart materials will be called *active optimal design*. Some recent advances in a smart material design are summarized in Anderson and Tsou (1992).

Examples of smart materials are:

(i) shape memory alloys;
(ii) composites featuring electrorheological fluids that possess the ability to alter their viscous properties under the presence of an electric field;
(iii) components made by embedding piezoceramic actuators and sensors into a matrix.

With ordinary (non-smart) materials, the design is necessarily *passive* and restrained. Indeed, if the goal is to create a material able to feature a desired response to a variety of external conditions which may be competitive or even contradictory, then structural resources will split to meet the set of requirements rather than to become entirely focused on the immediate environmental demand.

The concept of smart (active) material design is entirely different. Here, the system responds at each time to the immediate demand required by the environment. The effectiveness and flexibility of this design strategy is incomparatively greater than those of the passive approach. Conceptually speaking, the smart materials allow the designer to restrict himself with single purpose design instead of multi-purpose design. The structure will resemble a living tissue insofar as it is able to mobilize its properties to meet the immediate environmental demands. Another benefit associated with smartness is the ability to manipulate several physical properties independently.

The concept of active design is implemented through the assumed dependence of controls both on space and time. In the context of structure mechanics, this concept has so far been addressed with respect to *external loads* viewed as controls.

Probably the earliest work on this subject has been initiated by Butkovsky *et al.* (see Butkovsky *et al.* (1980), summarized in Butkovsky (1982). These authors introduced what they called "movable controls" to effectively suppress vibrations. Eventually, a similar concept has been exercised in McLaughlin and Slemrod (1986). A moving ("scanning") control is the one concentrated within a certain spatial domain, this domain moving through the space as a solid body. This idea works very well when applied to external sources (loads), it must work even better when implemented toward *material* or *structural parameters*.

The elastic moduli (stiffnesses) of structures are known to be very effective controllers. By now, there is plenty of information concerning optimal layout of materials differing in their elastic stiffnesses. This information is related either to statics, or to problems of free vibrations or loss of static stability, in all cases are materials assumed to be passive, i.e., their stiffnesses never being time dependent.

The concept of general space and time dependent stiffnesses is intrinsic in the idea of smart materials, and it promises substantially new possibilities of controlling the systems' behavior. Specifically, with such stiffnesses combined with space and time dependent material density, it becomes possible to selectively screen large domains in space-time from unwanted disturbances which could be initiated either by external influences (i.e., impacts), or by permanently applied loads. The following examples will illustrate this idea mathematically.

## 3. CONTROL OF THE FIRST ORDER EVOLUTION EQUATION

Consider a controlled system governed by the first order equation (Lurie (1971))

$$z_t + uz_x = 0, \quad 0 \leqslant t < \infty, \quad -\infty < x < \infty, \tag{1}$$

with $u = u(x, t)$ treated as control.

Complemented by the initial condition

$$z(0, x) = z_0(x), \quad -\infty < x < \infty, \tag{2}$$

this equation describes signals propagating with the phase velocity $u = u(x, t)$.

Suppose that $u = u(x, t)$ takes the values $u_1$, $u_2 (u_1 < u_2)$ as shown in Fig. 1:

$$u = \begin{cases} u_1 \text{ in parallelogram } abcd, \\ u_2 \text{ otherwise.} \end{cases} \tag{3}$$

In other words, assume that the line segment $ab$, describing the zone within which the phase velocity takes the value $u_1$, is itself traveling as a solid body with uniform velocity $U$ (see Fig. 1). If this velocity (represented as a slope of the parallelogram $abcd$) satisfies $u_2 > U > u_1$, then the disturbances originating on the initial manifold concentrate on the leeward side, $ad$, of the parallelogram. This concentration occurs due to the fact that the

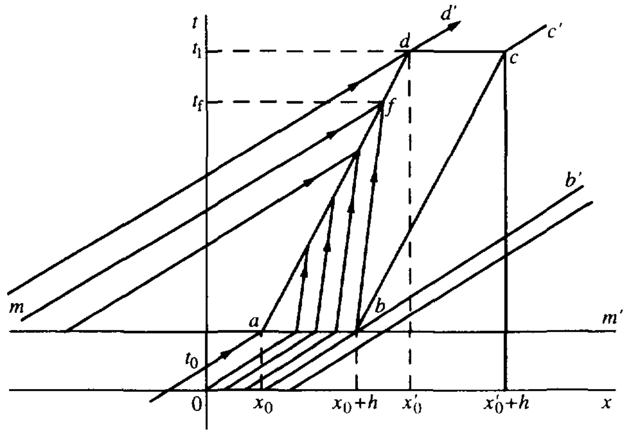


Fig. 1. The screening effect for the 1st order evolution equation.

disturbances initiated to the left of the segment *ad* overtake this segment in their subsequent travel. At the same time, the disturbances within the segment lag behind since the segment moves faster. Mathematically, this is expressed as the collision of characteristics arriving at *ad* from both sides. Because the array $(x_0, x_0+h)$ of oncoming disturbances has been redirected, no disturbances will enter the quiescent region ("shadow zone") *d′dfbb′* shown on the figure. This region is thus completely screened from the encroachment of a disturbance $z(x, t)$ which is identically zero within it.

## 4. CONTROL OF THE SECOND ORDER HYPERBOLIC EQUATION

The longitudinal wave propagation along elastic bars is governed by the second order hyperbolic equation of the type

$$(\rho z_t)_t - (kz_x)_x = 0; \tag{4}$$

this equation provides complexities compared with its 1st order counterpart (1). The main reason is because eqn (4) introduces *two* phase velocities: $a = \sqrt{k/\rho}$ and $-a = -\sqrt{k/\rho}$, i.e., the waves traveling in opposite directions. Desiring to suppress both waves in a sense indicated above we should take actions aimed to create waves travelling *both in the same direction*.

Such waves may be arranged through the formation of composites with parameters that are variable both in space and time.

More specifically, we will assume that the material parameters $\rho$, $k$ address the following properties:

(i) they are both space and time dependent;

(ii) at each point $(x, t)$ the pair $(\rho, k)$ may take either the values $(\rho_1, k_1)$ or the values $(\rho_2, k_2)$;

(iii) these admissible values are taken within alternating layers having the slope $dx/dt = V$ so chosen as to ensure regular transition of continuous disturbance $z(x, t)$ across the interface from one layer to another. In other words, both kinematic and dynamic compatibility conditions will be observed across the interface.

Condition (iii) will be satisfied if we postulate the following relationship between the characteristic slopes $a_i = \sqrt{k_i/\rho_i}$, $i = 1, 2$, and $V$ (we assume that $a_2 \geqslant a_1$)

$$\frac{V^2 - a_1^2}{V^2 - a_2^2} \geqslant 0. \tag{5}$$

The procedure that will eventually produce the screening effect applied to eqn (4) will

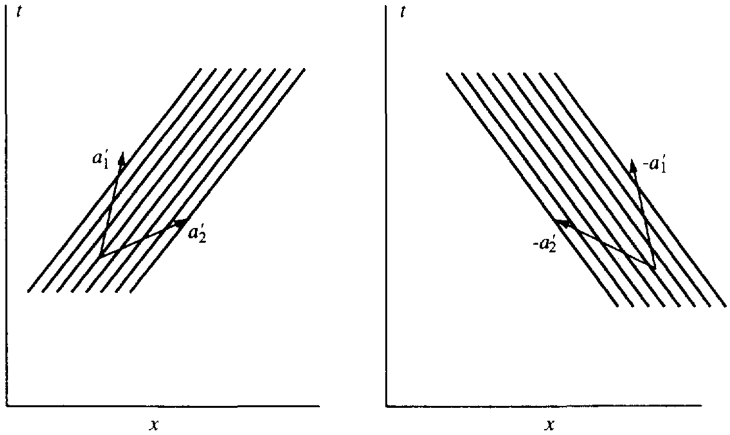


Fig. 2. (a) Composites of $r$-type. (b) Composites of $l$-type.

include two steps. First, we use the properties (i)–(iii) and choose the value of $V$ and the volume fraction $m_1$ of material $(\rho_1, k_1)$ in a composite to construct a rank 1 laminate with effective phase velocities $a_1'$, $a_2'$ $(a_2' \geqslant a_1')$ possessing *the same sign*, say positive, i.e., having disturbances traveling both to the right. We will call such a composite *r-composite* (Fig. 2a). Applying $-V$ instead of $V$ in the above construction, we create the *l-composite*, with effective phase velocities $-a_1'$, $-a_2'$ being both negative and related to waves traveling to the left (Fig. 2b).

At the second step we apply $r$- and $l$-composites to construct the material that implements the screening property. To this end assume that the $(x, t)$-plane is occupied by the $r$-composite everywhere but within the rectangle $mnpq$ where the $l$-composite is applied (Fig. 3). The characteristics of both families will then collide at the leeward face $mn$ of the rectangle, and the discontinuity will develop along it. At the same time, within the domain $rq\delta nu$ the value of $z$ will be identically zero because the initial disturbance $z(0, x)$ will never reach it since it will be averted from it on a halfway due to the appropriate turning of characteristic directions.

This screening effect will remain in place if we allow for the domain $mnpq$ to be a parallelogram with the slope $U$ of faces $mn$ and $pq$ satisfying the condition

$$|U| \leqslant \min(a_1', a_2').$$

As before, the screening effect has become possible due to the smartness of participating materials, specifically because of the time dependence of their stiffness and density. Both

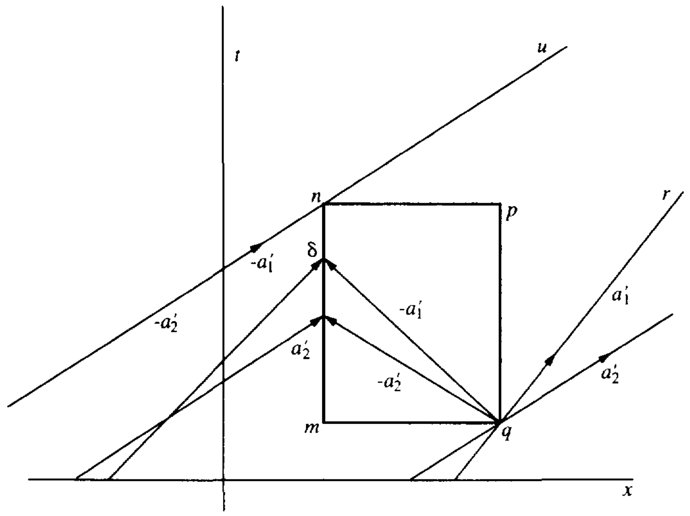


Fig. 3. The screening effect for the 2nd order hyperbolic equation.

characteristics may be made space and time dependent in a variety of ways, e.g., through the electrorheological fluid pumped in and sucked out along the bar at appropriate locations at appropriate time. The time dependence of density $\rho$ is intrinsic in the construction specified above: in the absence of such dependence this construction fails to work because the rank 1 composites mentioned above will then possess effective phase velocities of opposite signs. It is still unknown whether the screening effect will ever be possible lest the density be time dependent.

## 5. CONSTRUCTING THE $r$- AND $l$-COMPOSITES

We make use of conditions (i)–(iii) and inequality (5) to apply a standard procedure (Bakhvalov and Panasenko (1984), Lurie (1993)) of calculating the effective phase velocities of a composite assembled from the above mentioned layers distributed *in space-time*, i.e., on the $(x, t)$-plane.

Equation (4) is replaced by the system

$$\rho z^1_t = z^2_x, \quad k z^1_x = z^2_t \tag{6}$$

with $z^1 = z$, and $z^2$ denoting an auxiliary dependent variable. We will require that both $z^1$ and $z^2$ be continuous, thus ensuring the conditions of kinematic and dynamic compatibility, respectively, to hold. Formally, this means particularly that inequality (5) should be satisfied.

Let us introduce coordinates $\tau$, $n$, measured, respectively, as distances along and across the layers on the $(x, t)$-plane; the derivatives $z^1_t, \ldots, z^2_x$ may be expressed through $z^1_\tau, \ldots, z^2_n$ by the formulae

$$z^1_x = z^1_\tau x_\tau + z^1_n t_\tau, \quad z^1_t = z^1_\tau t_\tau - z^1_n x_\tau, \tag{7}$$

etc. The system (6) then takes on the form

$$\rho(z^1_\tau t_\tau - z^1_n x_\tau) = z^2_\tau x_\tau + z^2_n t_\tau,$$
$$k(z^1_\tau x_\tau + z^1_n t_\tau) = z^2_\tau t_\tau - z^2_n x_\tau,$$

or, equivalently,

$$z^1_n = \frac{\rho + k}{\rho x_\tau^2 - k t_\tau^2} z^1_\tau x_\tau t_\tau - \frac{1}{\rho x_\tau^2 - k t_\tau^2} z^2_\tau,$$
$$z^2_n = -\frac{\rho k}{\rho x_\tau^2 - k t_\tau^2} z^1_\tau + \frac{\rho + k}{\rho x_\tau^2 - k t_\tau^2} z^2_\tau x_\tau t_\tau. \tag{8}$$

Bearing in mind the continuity of $z^1$, $z^2$, i.e., that of $z^1_\tau$, $z^2_\tau$ across the layers' interface, we average both equations (8) over the layers. The quantities $z^1_\tau, \ldots, z^2_n$ will then be replaced by $\langle z^1 \rangle_\tau, \ldots, \langle z^2 \rangle_n$ where the operation $\langle \cdot \rangle = m_1(\cdot)_1 + m_2(\cdot)_2$ is expressed through the volume fractions $m_1, m_2 \geqslant 0$ $(m_1 + m_2 = 1)$ of materials 1 and 2 in the composite, and $(\cdot)_1$, $(\cdot)_2$ denote the values of $(\cdot)$ in these materials.

Preserving the notation $z^1_\tau, \ldots, z^2_n$ for $\langle z^1 \rangle_\tau, \ldots, \langle z^2 \rangle_n$, we arrive at the following system:

$$z^1_n = (A + B) z^1_\tau x_\tau t_\tau - C z^2_\tau,$$
$$z^2_n = -D z^1_\tau + (A + B) z^2_\tau x_\tau t_\tau, \tag{9}$$

with parameters $A, \ldots, D$ defined as

$$A = \left\langle \frac{k}{\rho x_\tau^2 - k t_\tau^2} \right\rangle, \quad B = \left\langle \frac{\rho}{\rho x_\tau^2 - k t_\tau^2} \right\rangle,$$

$$C = \left\langle \frac{1}{\rho x_\tau^2 - k t_\tau^2} \right\rangle, \quad D = \left\langle \frac{\rho k}{\rho x_\tau^2 - k t_\tau^2} \right\rangle. \tag{10}$$

By virtue of eqns (7), (10), the system (9) may be reduced to

$$\begin{aligned} \alpha z_x^1 t_\tau + \beta z_t^1 x_\tau &= z_x^2 x_\tau + z_t^2 t_\tau, \\ z_x^1 x_\tau + z_t^1 t_\tau &= \bar{\alpha} z_x^2 t_\tau + \bar{\beta} z_t^2 x_\tau, \end{aligned} \tag{11}$$

with

$$\alpha = A/C, \quad \beta = B/C, \quad \bar{\alpha} = A/D, \quad \bar{\beta} = B/D.$$

Evidently,

$$\frac{\bar{\alpha}}{\alpha} = \frac{\bar{\beta}}{\beta} = \Theta = \frac{C}{D},$$

and eqns (11) thus depend on three parameters: $\alpha$, $\beta$, $\Theta$. These parameters will be conveniently expressed through $\rho_i$, $k_i$ $(i = 1, 2)$, $m_1$, $m_2$, and $V = \mathrm{d}x/\mathrm{d}t$. To this end, we first notice that

$$A = \left\langle \frac{k}{\rho x_\tau^2 - k t_\tau^2} \right\rangle = \frac{m_1 k_1}{\rho_1 x_\tau^2 - k_1 t_\tau^2} + \frac{m_2 k_2}{\rho_2 x_\tau^2 - k_2 t_\tau^2} =$$

$$= \frac{k_1 k_2}{(\rho_1 x_\tau^2 - k_1 t_\tau^2)(\rho_2 x_\tau^2 - k_2 t_\tau^2)} \left[ \left( \widetilde{\frac{1}{a^2}} \right) x_\tau^2 - t_\tau^2 \right],$$

$$B = \left\langle \frac{\rho}{\rho x_\tau^2 - k t_\tau^2} \right\rangle = \frac{\rho_1 \rho_2}{(\rho_1 x_\tau^2 - k_1 t_\tau^2)(\rho_2 x_\tau^2 - k_2 t_\tau^2)} (x_\tau^2 - \tilde{a}^2 t_\tau^2),$$

$$C = \left\langle \frac{1}{\rho x_\tau^2 - k t_\tau^2} \right\rangle = \frac{1}{(\rho_1 x_\tau^2 - k_1 t_\tau^2)(\rho_2 x_\tau^2 - k_2 t_\tau^2)} (\tilde{\rho} x_\tau^2 - \tilde{k} t_\tau^2),$$

$$D = \left\langle \frac{\rho k}{\rho x_\tau^2 - k t_\tau^2} \right\rangle = \frac{1}{(\rho_1 x_\tau^2 - k_1 t_\tau^2)(\rho_2 x_\tau^2 - k_2 t_\tau^2)} [\rho_1 \rho_2 \langle k \rangle x_\tau^2 - k_1 k_2 \langle \rho \rangle t_\tau^2].$$

Here, $\tilde{a}^2 = m_1 a_2^2 + m_2 a_1^2$, $\langle k \rangle = m_1 k_1 + m_2 k_2$, etc. Parameters $\alpha$, $\beta$, $\Theta$ will be expressed as

$$\alpha = \frac{A}{C} = k_1 k_2 \frac{\left( \widetilde{\frac{1}{a^2}} \right) V^2 - 1}{\tilde{\rho} V^2 - \tilde{k}},$$

$$\beta = \frac{B}{C} = \rho_1 \rho_2 \frac{V^2 - \tilde{a}^2}{\tilde{\rho} V^2 - \tilde{k}},$$

$$\Theta = \frac{C}{D} = \frac{\tilde{\rho} V^2 - \tilde{k}}{\rho_1 \rho_2 \langle k \rangle V^2 - k_1 k_2 \langle \rho \rangle}. \tag{12}$$

Consider the characteristic form

$$z^1_{,xx}(x_\tau^2-\Theta\alpha^2 t_\tau^2)+z^1_{,tt}(t_\tau^2-\Theta\beta^2 x_\tau^2)+2z^1_{,xt}(1-\Theta\alpha\beta)x_\tau t_\tau$$

associated with eqns (11). The characteristics of (11) have the slopes on the $(x, t)$-plane computed as the roots $\lambda_1$, $\lambda_2$ of

$$\lambda^2(t_\tau^2-\Theta\beta^2 x_\tau^2)-2\lambda(1-\Theta\alpha\beta)x_\tau t_\tau+x_\tau^2-\Theta\alpha^2 t_\tau^2=0,$$

or, with the notation

$$V=\frac{dx}{dt},$$

$$\lambda^2(1-V^2\Theta\beta^2)-2\lambda V(1-\Theta\alpha\beta)+V^2-\Theta\alpha^2=0. \tag{13}$$

This equation has the roots

$$\lambda_1=-\frac{V-\alpha\sqrt{\Theta}}{V\beta\sqrt{\Theta}-1},\quad \lambda_2=\frac{V+\alpha\sqrt{\Theta}}{V\beta\sqrt{\Theta}+1}$$

with the product

$$\lambda_1\lambda_2=\frac{V^2-\Theta\alpha^2}{1-V^2\Theta\beta^2}.$$

The roots $\lambda_1$, $\lambda_2$ will be real if $\Theta>0$. The latter condition will be satisfied as long as inequality (5) holds. If $V=0$, then $\alpha=k_1k_2/\tilde{k}$, $\beta=\rho_1\rho_2\tilde{a}^2/\tilde{k}$, $\Theta=\tilde{k}/k_1k_2\langle\rho\rangle$, and $\lambda_1\lambda_2=-\Theta\alpha^2=-\langle\rho\rangle^{-1}\langle 1/\mathrm{k}\rangle^{-1}<0$. If $V=\infty$, then $\lambda_1\lambda_2=-1/\Theta\beta^2=-\langle k\rangle\langle 1/\rho\rangle<0$. We will investigate $\lambda_1\lambda_2$ as the function of $V^2$.

Referring to (12) we show that the following formulae hold:

$$V^2-\Theta\alpha^2=\frac{1}{\Delta}P,\quad 1-V^2\Theta\beta^2=\frac{1}{\Delta}Q,\quad 1-\Theta\alpha\beta=\frac{1}{\Delta}S$$

with

$$P=P(V)=(\tilde{\rho}V^2-\tilde{k})\left[\left(\widetilde{\frac{1}{k}}\right)V^2-\left(\widetilde{\frac{1}{\rho}}\right)\right]V^2-a_1^2a_2^2\left[\left(\widetilde{\frac{1}{a^2}}\right)V^2-1\right]^2,$$

$$Q=Q(V)=(\tilde{\rho}V^2-\tilde{k})\left[\left(\widetilde{\frac{1}{k}}\right)V^2-\left(\widetilde{\frac{1}{\rho}}\right)\right]-\frac{V^2}{a_1^2a_2^2}(V^2-\tilde{a}^2)^2,$$

$$S=S(V)=(\tilde{\rho}V^2-\tilde{k})\left[\left(\widetilde{\frac{1}{k}}\right)V^2-\left(\widetilde{\frac{1}{\rho}}\right)\right]-(V^2-\tilde{a}^2)\left[\left(\widetilde{\frac{1}{a^2}}\right)V^2-1\right],$$

$$\Delta=\Delta(V)=(\tilde{\rho}V^2-\tilde{k})\left[\left(\widetilde{\frac{1}{k}}\right)V^2-\left(\widetilde{\frac{1}{\rho}}\right)\right].$$

For $P$, $Q$, $S$ we have alternative expressions

$$P = \tilde{\rho}\left(\frac{\tilde{1}}{k}\right)\left[V^2 - \frac{1}{\tilde{\rho}\left(\frac{\tilde{1}}{k}\right)}\right](V^2 - a_1^2)(V^2 - a_2^2),$$

$$Q = -\frac{1}{a_1^2 a_2^2}\left[V^2 - \tilde{k}\left(\frac{\tilde{1}}{\rho}\right)\right](V^2 - a_1^2)(V^2 - a_2^2),$$

$$S = \left[\tilde{\rho}\left(\frac{\tilde{1}}{k}\right) - \left(\frac{\tilde{1}}{a^2}\right)\right](V^2 - a_1^2)(V^2 - a_2^2).$$

Equation (13) now obtains the form

$$Q\lambda^2 - 2VS\lambda + P = 0,$$

or, equivalently,

$$-\frac{1}{a_1^2 a_2^2}\left[V^2 - \tilde{k}\left(\frac{\tilde{1}}{\rho}\right)\right]\lambda^2 - 2V\left[\tilde{\rho}\left(\frac{\tilde{1}}{k}\right) - \left(\frac{\tilde{1}}{a^2}\right)\right]\lambda + \tilde{\rho}\left(\frac{\tilde{1}}{k}\right)\left[V^2 - \frac{1}{\tilde{\rho}\left(\frac{\tilde{1}}{k}\right)}\right] = 0.$$

The product $\lambda_1\lambda_2$ of its roots equals to

$$\lambda_1\lambda_2 = -\tilde{\rho}\left(\frac{\tilde{1}}{k}\right)a_1^2 a_2^2 \frac{V^2 - \dfrac{1}{\tilde{\rho}\left(\frac{\tilde{1}}{k}\right)}}{V^2 - \tilde{k}\left(\frac{\tilde{1}}{\rho}\right)}.$$

By direct inspection we show that ($\Delta\rho = \rho_2 - \rho_1$, etc.)

$$\tilde{k}\tilde{\rho}\left(\frac{\tilde{1}}{\rho}\right)\left(\frac{\tilde{1}}{k}\right) = \left(1 + \frac{m_1 m_2}{\rho_1\rho_2}(\Delta\rho)^2\right)\left(1 + \frac{m_1 m_2}{k_1 k_2}(\Delta k)^2\right) > 1,$$

and, consequently,

$$\tilde{k}\left(\frac{\tilde{1}}{\rho}\right) > \frac{1}{\tilde{\rho}\left(\frac{\tilde{1}}{k}\right)}. \tag{14}$$

If $V^2$ is so chosen that

$$\tilde{k}\left(\frac{\tilde{1}}{\rho}\right) > V^2 > \frac{1}{\tilde{\rho}\left(\frac{\tilde{1}}{k}\right)}, \tag{15}$$

then $\lambda_1\lambda_2 > 0$. The right inequality (15) will become consistent with (5) if we show that the inequality

$$\frac{1}{\tilde{\rho}\left\langle\frac{1}{k}\right\rangle} < a_1^2 \tag{16}$$

is possible, and then choose $V^2$ satisfying

$$\frac{1}{\tilde{\rho}\left\langle\frac{1}{k}\right\rangle} < V^2 < a_1^2 < a_2^2.$$

Inequality (16) is confirmed by direct calculation:

$$L = \frac{1}{\tilde{\rho}\left\langle\frac{1}{k}\right\rangle} - a_1^2 = \frac{k_1 k_2}{\tilde{\rho}\langle k\rangle} - \frac{k_1}{\rho_1} = \frac{k_1}{\rho_1 \tilde{\rho}\langle k\rangle}[k_2\rho_1 - (m_1 k_1 + m_2 k_2)(m_1\rho_2 + m_2\rho_1)]$$

$$= \frac{m_1 k_1}{\rho_1 \tilde{\rho}\langle k\rangle}[(1+m_2)k_2\rho_1 - (1-m_2)k_1\rho_2 - m_2(k_2\rho_2 + k_1\rho_1)]$$

$$= \frac{m_1 k_1}{\rho_1 \tilde{\rho}\langle k\rangle}[\rho_1\Delta k - k_1\Delta\rho - m_2\Delta k\Delta\rho].$$

This expression can be made negative even though we assume that $a_2^2 > a_1^2$, i.e.,

$$a_2^2 - a_1^2 = \frac{k_2}{\rho_2} - \frac{k_1}{\rho_1} = \frac{1}{\rho_1\rho_2}(\rho_1\Delta k - k_1\Delta\rho) > 0.$$

To make $L$ negative, it is necessary that $\Delta k\Delta\rho > \rho_1\Delta k - k_1\Delta\rho > 0$. For example, if $k_2 = 10$, $\rho_2 = 9$, $k_1 = \rho_1 = 1$, then $\rho_1\Delta k - k_1\Delta\rho - m_2\Delta k\Delta\rho = 9 - 8 - 72m_2$, and this expression will be $\leqslant 0$ if $m_2 \geqslant 1/72$. At the same time, the difference $k_2/\rho_2 - k_1/\rho_1$ should be positive, i.e., the ratio $k/\rho$ should increase. Combined with $\Delta k\Delta\rho > 0$ this means that the increase may be due to that in $k$ and less intensive increase (not decrease) in $\rho$, or due to the decrease in $\rho$ and less intensive decrease (not increase) in $k$.

In observance of $L \leqslant 0$ and inequality (15), the product $\lambda_1\lambda_2 \geqslant 0$. We thus obtain laminates for which both effective phase velocities show the same sign. This sign can be changed to opposite if we switch from $V$ to $-V$. Both velocities will be positive for $r$-laminates and negative for $l$-laminates (Figs 2a, 2b).

*Remark* 1. Inequality (16) will become impossible if we assume that $\rho_1 = \rho_2$. Then

$$\frac{1}{\rho_1\left\langle\frac{1}{k}\right\rangle} - \frac{k_1}{\rho_1} = \frac{m_1}{\rho_1\left\langle\frac{1}{k}\right\rangle}\left(1 - \frac{k_1}{k_2}\right) > 0$$

since $k_2 > k_1$. In this case, the product $\lambda_1\lambda_2$ cannot be made positive. A similar result holds true when $k_1 = k_2$. We conclude that the roots of the same sign can be produced through a simultaneous variation of both density and stiffness.

*Remark* 2. Inequality (16) cannot be replaced by a stronger inequality

$$\tilde{k}\left(\frac{\bar{1}}{\rho}\right) < a_1^2.$$

This is impossible since (recall that $\Delta a^2 = a_2^2 - a_1^2 > 0$)

$$\begin{aligned}
\tilde{k}\left(\frac{\bar{1}}{\rho}\right) - a_1^2 &= (m_1 k_2 + m_2 k_1)\left(\frac{m_1}{\rho_2} + \frac{m_2}{\rho_1}\right) - \frac{k_1}{\rho_1} \\
&= m_1\left[(1-m_2)\frac{k_2}{\rho_2} - (1+m_2)\frac{k_1}{\rho_1} + m_2\left(\frac{k_2}{\rho_1} + \frac{k_1}{\rho_2}\right)\right] \\
&= m_1\left[(1-m_2)a_2^2 - (1+m_2)a_1^2 + m_2\left(a_2^2\frac{\rho_2}{\rho_1} + a_1^2\frac{\rho_1}{\rho_2}\right)\right] \\
&= \frac{m_1}{\rho_1\rho_2}\{(a_2^2 - a_1^2)\rho_1\rho_2 - m_2[(a_2^2 + a_1^2)\rho_1\rho_2 - a_2^2\rho_2^2 - a_1^2\rho_1^2]\} \\
&= \frac{m_1}{\rho_1\rho_2}\{(\Delta a^2)\rho_1\rho_2 - m_2[a_2^2\rho_2(\rho_1 - \rho_2) - a_1^2\rho_1(\rho_1 - \rho_2)]\} \\
&= \frac{m_1}{\rho_1\rho_2}\{(\Delta a^2)\rho_1\rho_2 + m_2\Delta\rho[a_1^2\rho_2 - a_1^2\rho_1 + \rho_2(\Delta a^2)]\} \\
&= \frac{m_1}{\rho_1\rho_2}\{(\Delta a^2)\rho_2[\rho_1 + m_2\Delta\rho] + m_2 a_1^2(\Delta\rho)^2\} \\
&= \frac{m_1}{\rho_1\rho_2}\{(\Delta a^2)\rho_2\langle\rho\rangle + m_2 a_1^2(\Delta\rho)^2\} > 0.
\end{aligned}$$

*Remark* 3. The screening effect will be preserved if $\rho_2 = k_2$ and $\rho_1 = k_1$ but $\rho_1 \neq \rho_2$. Inequality (5) will then be trivially satisfied by any value of $V^2$ since $a_1^2 = a_2^2 = 1$. By inequality (14),

$$\tilde{\rho}\left(\frac{\bar{1}}{\rho}\right) > 1,$$

and inequality (15) specifies the interval of $V^2$ making the product $\lambda_1\lambda_2$ positive. This case is special because it eliminates limitations on the admissible choice of $V^2$.

## 6. THE ANALOGY WITH GAS DYNAMICS

The effect observed here presents a close analogy with supersonic gas dynamics where disturbances travel with the speed $a$ of sound relative to the particles and are at the same time transferred by the particles moving with their own material velocity $v$. Should $|v|$ be greater than $a$, then the resulting speeds $v \pm a$ of disturbances relative to the laboratory system will be both of the same sign. In our case, a similar phenomenon emerges due to a local redistribution of mass density $\rho$ as well as to the oscillations of the local stiffness $k$. These oscillations result in the effective decrease of phase velocity due to the successive reflections of waves from the layers' interfaces and the subsequent attenuation of the transient wave (Bakhvalov and Panasenko (1984)). Both effects work together toward the desired overall result allowing the disturbances to propagate in the same direction relative to the laboratory frame.

The collision of characteristics occurs both in gas dynamics and in a control problem. In gas dynamics, it creates the shock waves emerging in the flow in accordance with the non-linear equations of motion. Particularly, the relevant set of conservation laws (mass, momentum, energy) is observed across the shock waves.

In a control problem, the appearance of a strong discontinuity along the side *mn* of the rectangle *mnpq* on Fig. 3 cannot, however, be allowed from the viewpoint of the integrity of the construction. The damaging influence of this discontinuity can be avoided once the bar is originally cut at the location of this discontinuity. Both sides of the cut will then be exposed to their own independent displacements; the left half of the bar may be disregarded and taken away. There will be no reflected wave traveling back to the right from the cut: the energy of the initial state confined to the segment *mq* on Fig. 3 will be transformed into the work required to produce the relevant displacement of the left end of the bar over the time period $m\delta$.

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